2.1 – Solution Curves without a Solution

If $y = y(x)$ is a differentiable function, then $\frac{dy}{dx}$ gives the slope of a tangent line at a point.

We can learn about a DE $\frac{dy}{dx} = f(x,y)$ by observing segments of tangent lines, called lineal elements.

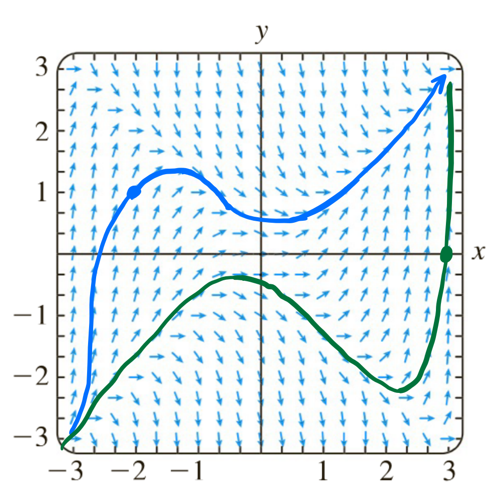
The collection of these lineal elements is called a direction field or a slope field.

In Problems 1–4 reproduce the given computer-generated direction field. Then sketch, by hand, an approximate solution curve that passes through each of the indicated points. Use different colored pencils for each solution curve.

1. $\frac{dy}{dx} = x^2 - y^2$

(a) $y(-2) = 1$ **(b)** $y(3) = 0$

(c) $y(0) = 2$ **(d)** $y(0) = 0$



In Problems 5–12 use computer software to obtain a direction field for the given differential equation. By hand, sketch an approximate solution curve passing through each of the given points.

5. $y' = x$

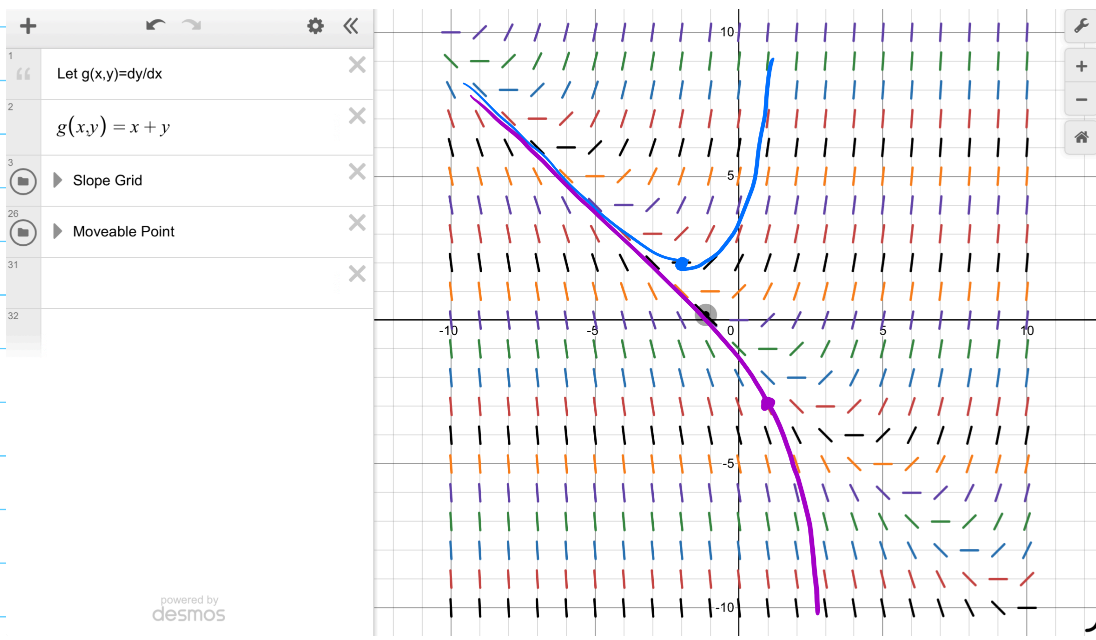
(a) $y(0) = 0$

(b) $y(0) = -3$

6. $y' = x + y$

(a) $y(-2) = 2$

(b) $y(1) = -3$



20. Consider the autonomous first-order differential equation $dy/dx = y^2 - y^4$ and the initial condition $y(0) = y_0$. By hand, sketch the graph of a typical solution $y(x)$ when y_0 has the given values.

(a) $y_0 > 1$

(b) $0 < y_0 < 1$

(c) $-1 < y_0 < 0$

(d) $y_0 < -1$

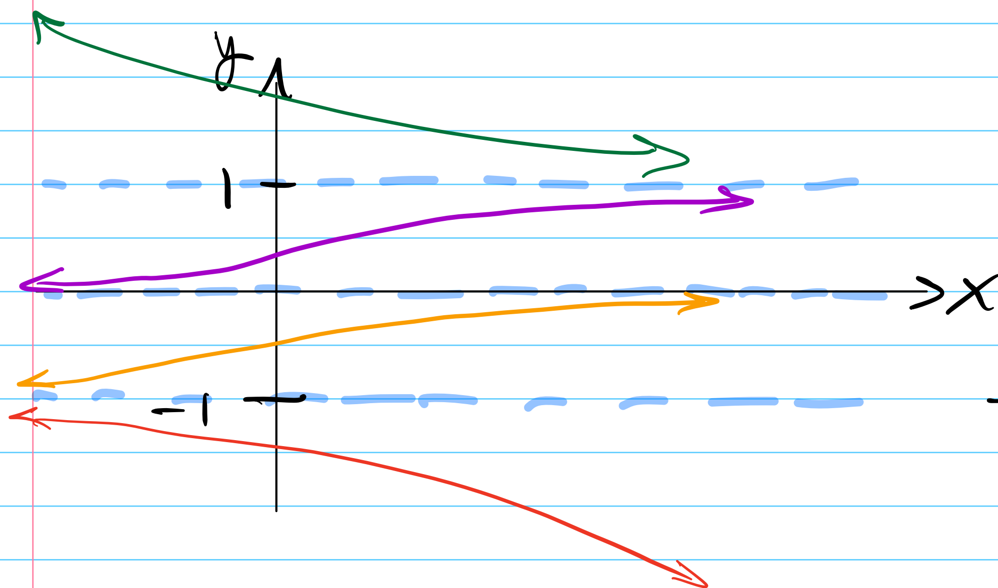
$y = 0,\ y = \pm 1$ are singular solutions

$$\frac{dy}{dx} = y^2 - y^4 = y^2(1+y)(1-y)$$

$$\frac{dy}{dx} = y^2(1+y)(1-y)$$

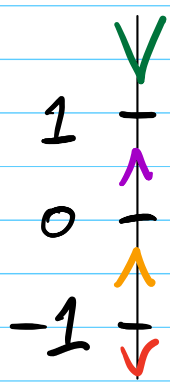
$$\frac{dy}{dx} = 0 \text{ if } y = -1, 0, 1$$

These are called critical numbers.



y	$\frac{dx}{dx}$
$y > 1$	neg
$y = 1$	0
$0 < y < 1$	pos
$y = 0$	0
$-1 < y < 0$	pos
$y = -1$	0
$y < -1$	neg

This information can be collapsed into a 1-D diagram, called a <u>phase portrait</u>.



1 is called an <u>attractor</u>

1 is asymptotically stable

0 is <u>semi-stable</u>

-1 is a <u>repeller</u>

-1 is unstable

$$\frac{dy}{y^2 - y^4} = dx$$

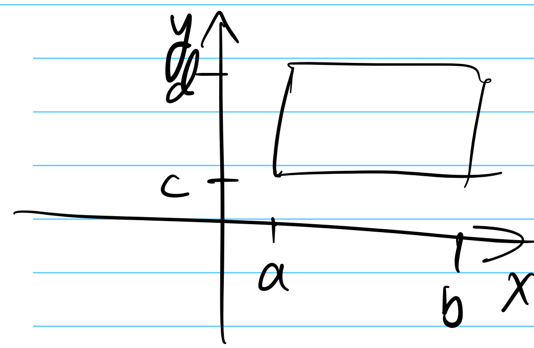
$y \neq 0, \pm 1$

THEOREM 1.2.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b$, $c \leq y \leq d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem (2).

Solve: $\quad \dfrac{dy}{dx} = f(x, y)$

Subject to: $\quad y(x_0) = y_0$



Consider $\quad y = \sin \underline{3} x^2$

$$y' = \underline{3} \cdot 2x \cos 3x^2$$

Now consider $\quad z = \sin y x^2$

$$\frac{\partial z}{\partial x} = y\, 2x \cos y x^2 \qquad \frac{\partial z}{\partial y} = x^2 \cos y x^2$$